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PROBLEM #1282

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Problem (1282)**.** *Find closed forms for the following constants. The products are over all primes p:*

$$\text{(a)} \prod_p \left(1+\frac{1}{p^2}\right)^{-1}$$

$$\text{(b)} \prod_p \left(1+\frac{1}{p^2}+\frac{1}{p^4}\right)^{-1}$$

$$\text{(c)} \prod_p \left(1+\frac{1}{p^2}+\frac{1}{p^4}+\frac{1}{p^6}\right)^{-1}$$

$$\text{(d)} \prod_p \left(1+\frac{1}{p^4}\right)^{-1}.$$

Solution. Define the function

$$\hat{\zeta}(n,k)=\prod_p \left(1+\frac{1}{p^k}+\left(\frac{1}{p^k}\right)^2+\cdots+\left(\frac{1}{p^k}\right)^n\right)^{-1},$$

for integers $n \geq 1$ and $k \geq 2$. By the formula for convergent geometric series we have

$$\hat{\zeta}(n,k)=\prod_p \left(\frac{1-\left(\frac{1}{p^k}\right)^{n+1}}{1-\frac{1}{p^k}}\right)^{-1}$$

(1)
$$=\prod_p \frac{\left(1-\frac{1}{p^{k(n+1)}}\right)^{-1}}{\left(1-\frac{1}{p^k}\right)^{-1}}.$$

Using the prime product form of the Euler-Riemann zeta function, $\zeta(s)=\prod_p (1-\frac{1}{p^s})^{-1}$ for $\text{Re}(s)>1$, the prime product of the numerator of (1) converges to $\zeta(k(n+1))$ and the prime product of the denominator converges to $\zeta(k)$. Thus, equation (1) converges, resulting in

(2)
$$\hat{\zeta}(n,k)=\frac{\zeta(k(n+1))}{\zeta(k)}.$$

Since k is even for (a), (b), (c), and (d), we can use Euler's formula $\zeta(2n)=(-1)^{n-1}\frac{B_{2n}}{2(2n)!}(2\pi)^{2n}$, where n is a natural number and B_i is the i-th Bernoulli number. In particular, we calculate the values $\zeta(2)=\frac{\pi^2}{6}$, $\zeta(4)=\frac{\pi^4}{90}$, $\zeta(6)=\frac{\pi^6}{945}$, and $\zeta(8)=\frac{\pi^8}{9450}$. We use these values and (2) to solve parts (a) through (d) as follows:

$$\begin{aligned}
&\text{(a)} \quad \prod_p \left(1+\frac{1}{p^2}\right)^{-1} &&= \hat{\zeta}(1,2) &&= \frac{\zeta(4)}{\zeta(2)} &&= \frac{\pi^2}{15}, \\
&\text{(b)} \quad \prod_p \left(1+\frac{1}{p^2}+\frac{1}{p^4}\right)^{-1} &&= \hat{\zeta}(2,2) &&= \frac{\zeta(6)}{\zeta(2)} &&= \frac{2\pi^4}{315}, \\
&\text{(c)} \quad \prod_p \left(1+\frac{1}{p^2}+\frac{1}{p^4}+\frac{1}{p^6}\right)^{-1} &&= \hat{\zeta}(3,2) &&= \frac{\zeta(8)}{\zeta(2)} &&= \frac{\pi^6}{1575}, \\
&\text{(d)} \quad \prod_p \left(1+\frac{1}{p^4}\right)^{-1} &&= \hat{\zeta}(1,4) &&= \frac{\zeta(8)}{\zeta(4)} &&= \frac{\pi^4}{105}.
\end{aligned}$$

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